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Axiomatic characterizations of the Walras correspondence for generalized economies

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Abstract

A generalized (pure exchange) economy consists of a list of agents, utility functions and initial endowments for each agent, and a net trade vector that describes the trade between the economy and the outside world. In this paper we extend the definition of Walras equilibria to generalized economies without free disposal of commodities. An existence result is presented and we provide two axiomatic characterizations of Walras allocations based on the axioms of consistency and converse consistency. Also, a characterization of the utility vectors corresponding to strictly positive Walras allocations is given. It is shown that the assumption of free disposal can, for generalized economies, lead to non-Pareto-optimal Walras allocations.

Keywords: Exchange economy; Walras allocation; Axiomatic characterization

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1. Introduction

The theory of competitive equilibria in exchange economies was first formulated in Walras (1874) and it has been studied extensively since. The existence of Walras equilibria is the subject of investigation in Arrow and Debreu (1954) (see

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also McKenzie, 1954), who provide an existence proof in the general case, and in Mas-Colell (1974), who provides an existence theorem without assuming completeness or transitivity of preferences. For a historical survey of the development of the theory of Walras equilibria the reader is referred to Arrow and Hahn (1971).

The goal of this paper is to axiomatize the Walras correspondence by means of its consistency properties. The consistency principle states that methods of reaching agreements should be consistent whatever the group of agents involved. More precisely, whenever the members in a decision-making situation have reached an agreement using some particular method, no subgroup of agents, given the acceptance of the complementary coalition and using the same method, has an incentive to reach a different agreement. The consistency principle unifies important developments in diverse areas ranging from abstract game theoretic models to concrete taxation and apportionment problems. Various solution concepts have been characterized on the basis of consistency properties. This line of research was initiated by Sobolev (1975), who axiomatized the prenucleolus, and it was followed by (among others) Peleg (1985, 1986) (axiomatizations of the core and the prekernel), Lensberg (1988) (axiomatization of the Nash bargaining solution), Hart and Mas-Colell (1989) (axiomatization of the Shapley value), Peleg and Tijs (1994), Norde et al. (1993), and Peleg et al. (1994) (axiomatizations of solutions of strategic games), van Heumen et al. (1994) (axiomatizations of solutions of Bayesian games), and by Peters et al. (1994) (axiomatizations of the Kalai–Smorodinsky and the egalitarian bargaining solutions). Some of the characterizations mentioned not only use a consistency axiom, but also an axiom of converse consistency. Converse consistency may be interpreted as a decentralization axiom, which states that a method of reaching agreements among the members of a decision-making situation should satisfy the following property: whenever a proposal is such that for all two-agent subgroups, given the acceptance of the complementary coalition, the corresponding proposal is in accordance with the decision-making method, then the proposal should qualify as an outcome of the decision-making process among the group of all agents of the decision-making situation. A survey on parts of the line of research described above is provided in Thomson (1990).

The existing attempts to characterize the Walras correspondence may be divided into three categories: the social choice approach, the indirect approach, and the direct approach.

In the social choice approach the initial endowments of the traders are fixed and, as a result, an exchange economy is completely determined by the profile of utilities of its members. Hence, the Walras correspondence can be viewed as a social choice correspondence and one can use suitable social choice axioms in order to obtain a characterization. Papers that follow this approach are Nagahisa (1991, 1992, 1994), Nagahisa and Suh (1993), and Gevers (1986). In the current paper, we do not fix the initial bundles and therefore our approach cannot be placed in this category.

In the indirect approach one attempts to characterize the Walras correspondence by comparing it with other well-known correspondences or by its implementability properties. For example, Dagan (1994) characterizes the Walras correspondence as a subcorrespondence of the core that has some additional properties, including consistency and a form of converse consistency. His proof uses the Debreu and Scarf (1963) limit theorem. For a comprehensive list of papers that follow the indirect approach we refer to Thomson (1988, Section 5).

The third category consists of papers that deal with the specific properties of the competitive equilibrium. Two examples are the Thomson (1988, 1992) papers. The first paper, however, deals with competitive equilibria from equal initial bundles. The second paper is an exploratory study and discusses consistency for exchange economies. It mainly contains ideas and definitions, but no theorems. Hence, our results are not covered by the papers of Thomson.

Our approach can be classified as a direct one. In order to deal with a variable number of agents we use the concept of generalized economies due to Thomson (1992) (see also Dagan, 1994). A generalized (pure exchange) economy is a list of agents, utility functions and initial endowments for each agent, and a net trade vector that describes the trade between the economy and the outside world. We focus on Walras allocations; allocations of the commodities available that are part of a Walras equilibrium. In order to guarantee the existence of a Walras equilibrium when there is no trade with the outside world (i.e. the net trade vector equals 0) we make the standard assumptions that the initial endowments are strictly positive and that the utility functions are continuous, quasi-concave, and satisfy local non-satiation. In order to prove converse consistency of the Walras correspondence, we also assume that the utility functions are smooth. Hence, the scope of this paper is limited to generalized economies with smooth utility functions. An axiomatic characterization of the Walras correspondence for economies with more general utility functions might require a different approach from the one we use.

The paper is organized as follows. In Section 2, we formally introduce generalized economies. Then, in Section 3, we prove that under the assumption that there is no free disposal of commodities, the Walras correspondence is the unique solution that satisfies consistency, converse consistency, Pareto optimality for two-agent economies, and necessity of a medium of exchange. The necessity of a medium of exchange axiom is a price-oriented exchange axiom. It states that, for one-agent economies, the agent in the economy maximizes his utility, given his budget set with respect to some price vector. Here, the price vector has to be such that the supply of commodities is equal to the demand for commodities. We show that this axiom can be replaced by non-emptiness for one-agent economies if we restrict ourselves to the family of generalized economies that have at least one Walras allocation. Section 4 contains an existence result for Walras allocations. In this section, we provide sufficient conditions on generalized economies such that there exists at least one Walras allocation. In Section 5, we show that the axioms consistency, Pareto optimality, and non-emptiness suffice to characterize the utility vectors corresponding to strictly positive Walras allocations. In Section 6, we briefly consider how the analysis changes if there is free disposal of commodities, and, finally, in Section 7, we put forward some ideas for future research.

2. Generalized economies

In this section we introduce some notations and definitions.

By $\mathbb{R}^{\ell}_{+}$ we denote the non-negative orthant of $\mathbb{R}^{\ell}$ and the strictly positive orthant of $\mathbb{R}^{\ell}$ is denoted by $\mathbb{R}^{\ell}_{++}$. Furthermore, for two vectors x, $y \in \mathbb{R}^{\ell}$ we denote $x \geqslant y$ $(x > y)$ if $x - y \in \mathbb{R}^{\ell}_{+}$ $(x - y \in \mathbb{R}^{\ell}_{++})$. The inner product xy of two vectors x and $y \in \mathbb{R}^{\ell}_{+}$ is defined by $xy = \sum_{i=1}^{\ell} x_i y_i$.

A *generalized economy* is a list $E = \langle N; (w^i)_{i \in N}; (u^i)_{i \in N}; \Theta \rangle$, where N (sometimes denoted $N(E)$) is a non-empty finite set of agents, $w^i \in \mathbb{R}^{\ell}_{++}$ is the strictly positive initial endowment of agent $i \in N$, u^i: $\mathbb{R}^{\ell}_{+} \to \mathbb{R}$ is the utility function of agent $i \in N$, and $\Theta \in \mathbb{R}^{\ell}$ is the net trade vector for the economy, satisfying $\sum_{i \in N} w^i + \Theta \geqslant 0$. The commodity space is $\mathbb{R}^{\ell}_{+}$, the non-negative orthant of $\mathbb{R}^{\ell}$. The net trade vector Θ indicates how much of each good is imported in the economy and, hence, a good that is exported from the economy will have a negative sign in the vector Θ. We will assume throughout the paper that the utility functions of the agents are continuous, quasi-concave and locally non-satiated. A utility function u: $\mathbb{R}^{\ell}_{+} \to \mathbb{R}$ is called locally non-satiated if for every commodity bundle $x \in \mathbb{R}^{\ell}_{+}$ and every open neighbourhood of x there exists a $y \in \mathbb{R}^{\ell}_{+}$ that is in this neighbourhood and that has a strictly higher utility than x $(u(y) > u(x))$. The class of all generalized economies with continuous, quasi-concave and locally non-satiated utility functions is denoted by $\mathscr{E}$.

In this paper, we opt for discussing economies in which there can be no free disposal of commodities. In a concluding section, we will briefly discuss the influence of the possibility of free disposal of commodities on our results.

An *allocation* in a generalized economy is an assignment of resources to the agents in the economy that is a redistribution of the resources that are available in the economy. Formally, denoting $A(E)$ for the set of allocations in economy E, we have

$$A(E) = \left\{ (x^i)_{i \in N} \mid x^i \in \mathbb{R}^{l}_{+} \text{ for each } i \in N \text{ and } \sum_{i \in N} x^i = \sum_{i \in N} w^i + \Theta \right\},$$

for each $E \in \mathscr{E}$. An allocation $x = (x^i)_{i \in N} \in A(E)$ is *efficient* or *Pareto optimal* for E if there is no other allocation $(y^i)_{i \in N} \in A(E)$ such that $u^i(y^i) \geqslant u^i(x^i)$ for all $i \in N$, with strict inequality for at least one $i \in N$. It is easy to prove that efficient allocations exist since the utility functions of the agents are continuous and the set of allocations is compact.

We will explain why a net trade vector Θ appears in our economies on the basis of the concept of a reduced economy, which is important when studying consistency (see Section 3). Let $E = \langle N; (w^i)_{i \in N}; (u^i)_{i \in N}; \Theta \rangle \in \mathscr{E}$ be a generalized economy. Let $S \subseteq N$, $S \neq \emptyset$, and let $x \in A(E)$. The *reduced economy* of E with respect to S and x is

$$E^{S,x} = \left\langle S; (w^i)_{i \in S}; (u^i)_{i \in S}; \Theta + \sum_{i \in N \setminus S} (w^i - x^i) \right\rangle.$$

The interpretation of the reduced economy $E^{S,x}$ is the following: suppose the agents in N agree on the allocation x and the agents in $N \setminus S$ leave the economy with their respective allocated bundles. Then the agents in S can reconsider their situation, only now they are left with a net trade obligation of $\Theta + \sum_{i \in N \setminus S}(w^i - x^i)$ with the outside world. Hence, the agents is S are now part of the reduced economy $E^{S,x}$. Note that even if the original economy is an ordinary economy ($\Theta = 0$), then the reduced economy with respect to some subset of agents and some allocation will, in general, have a net trade vector $\Theta \neq 0$.

We will show that the reduced economy $E^{S,x}$ is, indeed, a generalized economy. Since E is a generalized economy and $(x^i)_{i \in N} \in A(E)$, we have $\sum_{i \in N} x^i = \sum_{i \in N} w^i + \Theta$ and, consequently, $\sum_{i \in S} x^i = \sum_{i \in S} w^i + \sum_{i \in N \setminus S}(w^i - x^i) + \Theta$. But $x^i \in \mathbb{R}^{\ell}_+$ for each $i \in N$, so $\sum_{i \in S} x^i \geqslant 0$ and, therefore,

$$\sum_{i \in S} w^i + \left(\Theta + \sum_{i \in N \setminus S} (w^i - x^i) \right) \geqslant 0.$$

Note that we implicitly proved that $(x^i)_{i \in N} \in A(E)$ implies that $(x^i)_{i \in S} \in A(E^{S,x})$. Furthermore, it is straightforward to check that, for $E \in \mathscr{E}$, $(x^i)_{i \in N} \in A(E)$ and $S \subseteq T \subseteq N(E)$, $S \neq \emptyset$, it holds that $E^{S,x} = (E^{T,x})^{S,x^T}$, where x^T denotes $(x^i)_{i \in T}$.

Let $\mathscr{E}_0 \subseteq \mathscr{E}$ be a family of generalized economies. A *solution* on $\mathscr{E}_0$ is a function ϕ, which assigns to each economy $E \in \mathscr{E}_0$ a subset $\phi(E)$ of $A(E)$. Examples of solutions on the class $\mathscr{E}$ of generalized economies are the *Pareto-optimal solution*, defined by $\mathrm{PO}(E) = \{x \in A(E) \mid x \text{ is efficient}\}$ for all $E \in \mathscr{E}$, and the *Walras correspondence*, defined by $\mathrm{W}(E) = \{(x^i)_{i \in N} \in A(E) \mid$ there exists a price vector $p \in \Delta$ such that $x^i \in D^i(p)$ for each $i \in N\}$. Here, $\Delta = \{p \in \mathbb{R}^{\ell}_+ \mid \sum_{j=1}^{\ell} p_j = 1\}$ and $D^i(p)$ is the *demand set of agent i with respect to the price vector p* defined by $D^i(p) := \{x^i \in \mathbb{R}^{\ell}_+ \mid px^i \leqslant pw^i$ and $u^i(x^i) \geqslant u^i(y)$ for all $y \in \mathbb{R}^{\ell}_+$ with $py \leqslant pw^i\}$.

Our definitions of the Pareto-optimal solution and the Walras correspondence for generalized economies are extensions of the usual definitions for ordinary economies ($\Theta = 0$). Using the same line of reasoning as for ordinary economies, it can be proved that $\mathrm{W}(E) \subseteq \mathrm{PO}(E)$ for all generalized economies $E \in \mathscr{E}$, where the inclusion is in general a strict inclusion. Further, we easily obtain the following result with respect to Walras allocations.

Proposition 2.1. Let $E = \langle N; (w^i)_{i \in N}; (u^i)_{i \in N}; \Theta \rangle$ be a generalized economy, let x be a Walras allocation for E ($x \in \mathrm{W}(E)$), and let $p \in \Delta$ be a price vector such that $x^i \in D^i(p)$ for each $i \in N$. Then it holds that $p\Theta = 0$.

Proof. Because the utility functions of the agents are locally non-satiated, it holds that $px^i = pw^i$ for each $i \in N$. Hence, $\Sigma_{i \in N} px^i = \Sigma_{i \in N} pw^i$. Further, because $x \in A(E)$, we have $\Sigma_{i \in N} x^i = \Sigma_{i \in N} w^i + \Theta$ and, consequently, $p\Sigma_{i \in N} x^i = p(\Sigma_{i \in N} w^i + \Theta)$. Now, the proposition follows by substituting $\Sigma_{i \in N} pw^i$ for $p\Sigma_{i \in N} x^i$. □

Proposition 2.1 implies that, given the agent's initial endowments and utility functions, the Walras correspondence may be empty for many net trade vectors. However, see Remark 3.2 below on the non-emptiness of the Walras correspondence.

3. Characterizations of the Walras correspondence

In this section we will provide two axiomatic characterizations of the Walras correspondence based on the axioms of consistency and converse consistency. Both axioms are defined using the reduced economies introduced in the previous section.

A solution ϕ on a family of economies $\mathscr{E}_0$ is *consistent* (CONS) if it satisfies the following condition. If $E \in \mathscr{E}_0$, $S \subseteq N(E)$, $S \neq \emptyset$, and $x \in \phi(E)$, then $E^{S,x} \in \mathscr{E}_0$ and $x^S \in \phi(E^{S,x})$. Our notion of consistency is only slightly different from Dagan's (1994) notion. The difference is that Dagan puts $E^{S,x} \in \mathscr{E}_0$ as a condition, whereas we put it as a consequence. Therefore, Dagan's notion seems to be weaker at first sight. However, for all families $\mathscr{E}_0 \subseteq \mathscr{E}$ that we consider in this paper it holds that $E^{S,x} \in \mathscr{E}_0$ for all $E \in \mathscr{E}_0$, $S \subseteq N$, $S \neq \emptyset$, and all $x \in A(E)$, so in the context of this paper the two definitions are equivalent. The following lemma was also proved in Dagan (1994).

Lemma 3.1. The Walras correspondence on the class $\mathscr{E}$ is consistent.

Proof. Let E be a generalized economy, let $S \subseteq N(E)$, $S \neq \emptyset$, and let $x \in W(E)$. We have already seen that $E^{S,x} \in \mathscr{E}$. Furthermore, there exists a price vector $p \in \Delta$ such that $x^i \in D^i(p)$ for each $i \in N$. Since $x \in A(E)$, we know that $x^S \in A(E^{S,x})$. Hence, using the fact that $x^i \in D^i(p)$ for each $i \in S$, it follows that $x^S \in W(E^{S,x})$. □

Remark 3.2. Proposition 2.1 implies that the Walras correspondence will often be empty. However, it follows by Lemma 3.1 and the existence of Walras allocations for ordinary economies ($\Theta = 0$) that Walras allocations exist for all reduced economies of ordinary economies with respect to a Walras allocation.

Now, let us define converse consistency. A solution ϕ on a family of generalized economies $\mathscr{E}_0$ is *converse consistent* (COCONS) if for every $E \in \mathscr{E}_0$

with at least two agents ($|N(E)| \geqslant 2$), and for every $x \in A(E)$ that is efficient, the following condition is satisfied. If for every $S \subseteq N(E)$, $S \notin \{\emptyset, N(E)\}$, it holds that $E^{S,x} \in \mathscr{E}_0$ and $x^S \in \phi(E^{S,x})$, then $x \in \phi(E)$. Actually, COCONS is very similar to the converse consistency for the strong Nash equilibrium used in Peleg and Tijs (1994).

The Walras correspondence satisfies converse consistency if we put a smoothness condition on the utility functions. A utility function u: $\mathbb{R}^{\ell}_{+} \to \mathbb{R}$ is *smooth* if the following conditions are satisfied:

(i) If $x \in \mathbb{R}^{\ell}_{++}$, $y \in \mathbb{R}^{\ell}_{+}$, and $u(x) = u(y)$ then, $y \in \mathbb{R}^{\ell}_{++}$.

(ii) If $x \in \mathbb{R}^{\ell}_{++}$, then there exists a unique price vector $p \in \Delta$ such that $py \geqslant px$ for all $y \in G(x)$, where $G(x) = \{y \in \mathbb{R}^{\ell}_{+} \mid u(y) \geqslant u(x)\}$.

Note that conditions (i) and (ii) imply that $u(x) > u(y)$ for all $x \in \mathbb{R}^{\ell}_{++}$ and $y \in \mathbb{R}^{\ell}_{+} \setminus \mathbb{R}^{\ell}_{++}$. Smoothness also implies weak monotonicity, i.e. if a utility function u: $\mathbb{R}^{\ell}_{+} \to \mathbb{R}$ is smooth, then $u(y) \geqslant u(x)$ for all $x \in \mathbb{R}^{\ell}_{+}$ and $y \in \mathbb{R}^{\ell}_{++}$ with $y \geqslant x$. Furthermore, smoothness implies that for each generalized economy $E \in \mathscr{E}$ and each Walras allocation $x \in \mathrm{W}(E)$ there exists a unique price vector $p \in \Delta$ such that $x^i \in D^i(p)$ for each $i \in N(E)$.

Lemma 3.3. Let $E = \langle N; (w^i)_{i \in N}; (u^i)_{i \in N}; \Theta \rangle$ be such that u^i is smooth for each $i \in N$ and let x be a Walras allocation for E. Then there exists a unique price vector $p \in \Delta$ such that $x^i \in D^i(p)$ for each $i \in N$.

Proof. Obviously, there exists at least one price vector $p \in \Delta$ such that $x^i \in D^i(p)$ for each $i \in N$, by the definition of Walras allocations. To prove uniqueness, let $p \in \Delta$ be a price vector such that $x^i \in D^i(p)$ for each $i \in N$ and fix $j \in N$. Then, because u^j is locally non-satiated, $px^j = pw^j$. Also, if $y \in \mathbb{R}^{\ell}_{+}$ and $u^j(y) > u^j(x^j)$, then $py > pw^j = px^j$. Thus, using continuity of u^j, $py \geqslant px^j$ for all $y \in \mathbb{R}^{\ell}_{+}$ such that $u^j(y) \geqslant u^j(x^j)$. Further, $u^j(x^j) \geqslant u^j(w^j)$ and $w^j \in \mathbb{R}^{\ell}_{++}$. Hence, we know that $x^j \in \mathbb{R}^{\ell}_{++}$. Now, it follows from condition (ii) of smoothness that p is unique. □

Let $\mathscr{F}$ be the family of generalized economies where the agents have smooth utility functions. Note that within the family $\mathscr{F}$ we have enough assumptions to guarantee the validity of the two welfare theorems for ordinary economies. Hence, if $E \in \mathscr{F}$ has a net trade vector $\Theta = 0$, then the two welfare theorems are valid for this economy. Further, it is easily seen that for each economy $E \in \mathscr{F}$, each $S \subseteq N(E)$, $S \neq \emptyset$, and $x \in A(E)$, the reduced economy of E with respect to S and x is a member of the family $\mathscr{F}$. It follows that the Walras correspondence is consistent on $\mathscr{F}$.

The following lemma implies that the Walras correspondence is converse consistent on $\mathscr{F}$.

Lemma 3.4. Let $E = \langle N; (w^i)_{i \in N}; (u^i)_{i \in N}; \Theta \rangle \in \mathscr{F}$ and let $x \in A(E)$. Then x is a Walras allocation for E if and only if x is efficient and $x^i \in \mathrm{W}(E^{\{i\},x})$ for all $i \in N$.

Proof. Clearly, if $x \in \mathrm{W}(E)$ then x is efficient. Furthermore, by the consistency of the Walras correspondence we know that $x^i \in \mathrm{W}(E^{\{i\},x})$ for all $i \in N$. Assume now that x is efficient and that $x^i \in \mathrm{W}(E^{\{i\},x})$ for all $i \in N$. Then, for each $i \in N$, there exists a price vector p^i such that $x^i \in D^i(p^i)$. Following the proof of Lemma 3.3, we see that p^i is the unique price vector such that $x^i \in D^i(p^i)$.

Now, by our assumption, x is Pareto optimal in E. Clearly, this implies that x is Pareto optimal in the ordinary economy $\langle N; (x^i)_{i \in N}; (u^i)_{i \in N}; 0 \rangle$ (this property will reappear in Section 6 under the name self-efficiency and we refer to this section for an explanation of this property). Now, we can apply the second welfare theorem to the ordinary economy $\langle N; (x^i)_{i \in N}; (u^i)_{i \in N}; 0 \rangle$ and we conclude that there exists a $p \in \Delta$ that supports x^i for each $i \in N$. Therefore, by uniqueness of the prices p^i such that $x^i \in D^i(p^i)$ for each $i \in N$, we may conclude that $p^i = p$ for all $i \in N$. Thus, x is a Walras allocation for E. □

Corollary 3.5. The Walras correspondence is converse consistent on the family $\mathscr{F}$ of generalized economies.

Proof. Let $E = \langle N; (w^i)_{i \in N}; (u^i)_{i \in N}; \Theta \rangle \in \mathscr{F}$ with $|N| \geqslant 2$ and let $x \in A(E)$ be efficient and such that for every $S \subseteq N$, $S \notin \{\emptyset, N\}$, it holds that $x^S \in \mathrm{W}(E^{S,x})$. Then, obviously, $x^i \in \mathrm{W}(E^{\{i\},x})$ for all $i \in N$. Now, by Lemma 3.4, it follows that $x \in \mathrm{W}(E)$. □

Smoothness of the utility functions is a necessary condition in Corollary 3.5 in the sense that converse consistency may be violated if the utility functions are not smooth.

Dagan (1994) also proved that the Walras correspondence satisfies converse consistency. However, his notion of converse consistency is different from ours. The most important difference between the two notions is that the notion of Dagan (1994) can be applied only to generalized economies with more than two agents, whereas ours is also applicable for two-agent economies because we only consider efficient allocations.

In order to characterize the Walras correspondence we need two more axioms. A solution ϕ on a family of generalized economies $\mathscr{E}_0$ satisfies *Pareto optimality for two-agent generalized economies* (PO(2)) if for every two-agent generalized economy $E \in \mathscr{E}_0$ all $x \in \phi(E)$ are efficient. Furthermore, ϕ satisfies the *necessity of a medium of exchange for one-agent generalized economies* (NMEX(1)) if $\phi(E) = \mathrm{W}(E)$ for every one-agent generalized economy $E \in \mathscr{E}_0$. The necessity of a medium of exchange axiom can be interpreted as a price oriented exchange axiom that states that for one-agent economies the agent in the economy maximizes his utility given his budget set with respect to some price vector such that the supply of commodities is equal to the demand for commodities.

Using Lemma 3.4 and Corollary 3.5, we obtain the following result.

Theorem 3.6. The Walras correspondence is the unique solution on $\mathscr{F}$ that satisfies NMEX(1), *PO*(2), *CONS and COCONS.*

Proof. Clearly, the Walras correspondence satisfies NMEX(1) and PO(2). Furthermore, in Lemma 3.1 and Corollary 3.5 we proved that the Walras correspondence satisfies CONS and COCONS. Assume that ϕ is a solution on $\mathscr{F}$ that also satisfies the four foregoing axioms and let $E = \langle N; (w^i)_{i \in N}; (u^i)_{i \in N}; \Theta \rangle$ be in $\mathscr{F}$. If $|N| = 1$, then $\phi(E) = \mathrm{W}(E)$ by NMEX(1). If $|N| = 2$, let first $x \in \mathrm{W}(E)$. Then x is efficient and, by CONS of the Walras correspondence, $x^i \in \mathrm{W}(E^{\{i\},x}) = \phi(E^{\{i\},x})$ for all $i \in N$. So, by COCONS of ϕ we know that $x \in \phi(E)$. Now, let $x \in \phi(E)$. By PO(2), x is efficient and $x^i \in \phi(E^{\{i\},x}) = \mathrm{W}(E^{\{i\},x})$ for all $i \in N$ by CONS of ϕ. Hence, by Lemma 3.4, $x \in \mathrm{W}(E)$. We proceed by induction on the number of agents in an economy. Suppose that we proved that the Walras correspondence and ϕ coincide for all economies with, at most, k agents ($k \geqslant 2$). Now, suppose $|N| = k+1$. If $x \in \mathrm{W}(E)$, then x is efficient and, by the induction hypothesis, $x^S \in \mathrm{W}(E^{S,x}) = \phi(E^{S,x})$ for every $S \subseteq N$, $S \notin \{\emptyset, N\}$ and by COCONS of ϕ it holds that $x \in \phi(E)$. If $x \in \phi(E)$, then $x^S \in \phi(E^{S,x}) = \mathrm{W}(E^{S,x})$ for every $S \subseteq N$, $S \neq N$, by CONS of ϕ. Note that we cannot apply COCONS of the Walras correspondence, because we do not know whether x is efficient. Therefore, we proceed in the following way. For each $i \in N$ it holds that x^i is part of a Walras allocation in some reduced economy and, therefore, there is a price vector $p^i \in \Delta$ such that $x^i \in D^i(p^i)$. It follows from the proof of Lemma 3.3 that each p^i is unique. We claim that $p^i = p^j$ for any two agents i and $j \in N$, for if $p^i \neq p^j$, consider the reduced economy $E^{S,x}$ where $S = \{i, j\}$. Since $x^S \in \mathrm{W}(E^{S,x})$, there is some price vector p such that both $x^i \in D^i(p)$ and $x^j \in D^j(p)$. Then, either $p \neq p^i$ or $p \neq p^j$, which means that either x^i or x^j is supported by two different price vectors, contradicting unicity of the supporting price vectors. Now, we may conclude that there exists a price vector p such that $x^i \in D^i(p)$ for each $i \in N$. But this means that $x \in \mathrm{W}(E)$. □

Dagan (1994) also provides a characterization of the Walras correspondence using consistency and converse consistency. Besides the fact that his notion of converse consistency is different from ours, there is a more important difference between the two characterizations. Dagan (1994) needs a domain of economies that includes all economies with a finite number of agents (variable number of agents), whereas it follows from the proof of Theorem 3.6 that we only need a domain that satisfies the condition that for every economy in the domain all its reduced economies are also in the domain.

In our characterization, we can replace the 'necessity of a medium of exchange' axiom by a 'non-emptiness' axiom if we restrict the family of economies.

Let $\mathcal{G}$ be the family of generalized economies where agents have smooth utility functions and that allow for a Walras allocation, i.e. $\mathcal{G} = \{E \in \mathcal{F} \mid \mathrm{W}(E) \neq \emptyset\}$. A solution ϕ on $\mathcal{G}$ satisfies *non-emptiness for one-agent generalized economies* (NEM(1)) if $\phi(E) \neq \emptyset$ for every one-agent generalized economy $E \in \mathcal{G}$.

Theorem 3.7. The Walras correspondence is the unique solution on $\mathcal{G}$ that satisfies NEM(1), PO(2), CONS, and COCONS.

Proof. Clearly, the Walras correspondence satisfies NEM(1), PO(2), and COCONS on $\mathcal{G}$. Furthermore, it follows from CONS of the Walras correspondence on $\mathcal{F}$ that for each economy $E \in \mathcal{G}$, each $S \subseteq N(E)$, $S \neq \emptyset$, and $x \in \mathrm{W}(E)$, the reduced economy of E with respect to S and x is a member of the family $\mathcal{G}$. It follows that the Walras correspondence is consistent on $\mathcal{G}$.

Now, let ϕ be a solution on $\mathcal{G}$ that also satisfies the four axioms. If $E \in \mathcal{G}$ with $|N(E)| = 1$, then $|A(E)| = 1$ and, because both ϕ and the Walras correspondence satisfy NEM(1), $\phi(E) = \mathrm{W}(E) = A(E)$. This proves that ϕ satisfies NMEX(1) on $\mathcal{G}$. Now we can follow the proof of Theorem 3.6 and show that $\phi(E) = \mathrm{W}(E)$ for all $E \in \mathcal{G}$. □

Remark 3.8. The domain of the characterization in Theorem 3.7 is the set of all generalized economies with smooth utility functions for which the Walras correspondence is non-empty. In this way, the Walras correspondence enters into its own axiomatic characterization. Hence, Theorem 3.7 is much more a discovery of the most important properties of the Walras correspondence on a 'natural domain' than it is a justification of the Walras correspondence as a solution.

Remark 3.9. If ϕ is a solution on $\mathcal{G}$ that satisfies PO(2) and CONS, then $\phi(E) \subseteq \mathrm{W}(E)$ for all $E \in \mathcal{G}$.

This remark can easily be proved using the ideas in the proofs of Theorems 3.6 and 3.7.

Remark 3.10. The axiom COCONS is not redundant in the characterization in Theorem 3.7. This can be seen as follows. Let $E = \langle N; (w^i)_{i \in N}; (u^i)_{i \in N}; \Theta \rangle \in \mathcal{G}$. Then $i, j \in N$ are called *symmetric* if $w^i = w^j$ and $u^i = u^j$. An allocation $x \in A(E)$ satisfies *equal treatment* if $x^i = x^j$ whenever i and j are symmetric. Consider the equal treatment Walras correspondence $\mathrm{W}_e(E) = \{x \in \mathrm{W}(E) \mid x$ satisfies equal treatment$\}$. This correspondence satisfies NEM(1), PO(2), and CONS on $\mathcal{G}$. Furthermore, $\mathrm{W}_e(E) \subseteq \mathrm{W}(E)$, for each $E \in \mathcal{G}$, where the inclusion is, in general, a strict inclusion.

4. An existence result

In this section we provide a sufficient condition on generalized economies such that there exists at least one Walras allocation.

Let E be a generalized economy where the net trade vector Θ is such that it has at least one coordinate that is strictly positive and at least one coordinate that is strictly negative. This seems to be a reasonable assumption because the case $\Theta = 0$ is known and for a $\Theta \neq 0$ that is strictly positive or strictly negative we have non-existence when there is only one agent in the economy. We will compactify the consumption sets of the agents and provide conditions on the compactified economy that are sufficient for the existence of Walras allocations. Existence of Walras allocations for the generalized economy then follows by weak monotonicity of the utility functions.

Let $\rho \in \mathbb{R}_+$ and let E_ρ be the economy that is obtained from E when the consumption sets of the agents are replaced by the cube $Q_\rho = \{x \in \mathbb{R}_+^\ell \mid x_j \leqslant \rho$ for all $j \in \{1, \ldots, l\}\}$. Assume that ρ is large enough, so that $\Sigma_{i \in N} w^i$ and $\Sigma_{i \in N} w^i + \Theta$ are interior points of Q_ρ. For $p \in \Delta$ and $i \in N$, denote by $D_\rho^i(p)$ the demand set of agent i with respect to the price vector p, i.e. $D_\rho^i(p) := \{x^i \in Q_\rho \mid px^i \leqslant pw^i$ and $u^i(x^i) \geqslant u^i(y)$ for all $y \in Q_\rho$ with $py \leqslant pw^i\}$. For $p \in \Delta$ denote by $z(p) = \Sigma_{i \in N} D_\rho^i(p) - \Sigma_{i \in N} w^i - \Theta$, the excess demand correspondence of E_ρ at p. Also denote $L^+ = \{j \mid \Theta_j > 0\}$ and $L^- = \{j \mid \Theta_j < 0\}$. Note that both L^+ and L^- are non-empty. Furthermore, suppose that the following condition is satisfied: for all $p \in \Delta$ there exists a $\zeta \in z(p)$ such that $\zeta\Theta = 0$ and

$$\frac{\zeta_i}{\Theta_i} \leqslant \frac{\zeta_j}{\Theta_j},$$

for all $i \in L^-$ and all $j \in L^+$. This condition is rather strong, but this is not surprising because it is rarely satisfied that there exist Walras allocations if the net trade vector Θ is not equal to 0.

Theorem 4.1. If all the conditions above are satisfied, then E_ρ has a Walras allocation.

Proof. For $p \in \Delta$ let $z^*(p)$ be the set of all $\zeta \in z(p)$ that satisfy the conditions mentioned above, i.e. $z^*(p) = \{\zeta \in z(p) \mid \zeta\Theta = 0$ and $(\zeta_i/\Theta_i) \leqslant (\zeta_j/\Theta_j)$ for all $i \in L^-$ and $j \in L^+\}$. Then the multifunction z^* has non-empty and convex values and it is upper semi-continuous. Now, let $A \subseteq \mathbb{R}^\ell$ be such that $\bigcup_{p \in \Delta} z^*(p) \subseteq A$. We may choose A to be convex and compact. For $\zeta \in A$ we define

$$f(\zeta) = \{q \in \Delta_\Theta \mid q\zeta \geqslant \bar{q}\zeta \text{ for all } \bar{q} \in \Delta_\Theta\},$$

where $\Delta_\Theta = \{q \in \Delta \mid q\Theta = 0\}$. Then the multifunction f has non-empty convex values and it is upper semi-continuous.

Since both z^* and f are non-empty and convex-valued, upper-semi-continuous multifunctions, the multifunction h defined by $h(p, \zeta) = f(\zeta) \times z^*(p)$ for $(p, \zeta) \in \Delta \times A$, has a fixed point $(\bar{p}, \bar{\zeta}) \in f(\zeta) \times z^*(\bar{p})$. We will show that $\zeta = 0$, which implies that E_ρ has a Walras allocation.

Because $\bar{p} \in f(\bar{\zeta}) \subseteq \Delta_\Theta$, we know that $\bar{p}\Theta = 0$ and $\bar{p}\bar{\zeta} \geqslant q\bar{\zeta}$ for all $q \in \Delta_\Theta$. Furthermore, since $\bar{\zeta} \in z^*(\bar{p})$, we know by local non-satiation that $\bar{p}\bar{\zeta} = \bar{p}\Theta$. Hence, $0 \geqslant q\bar{\zeta}$ for all $q \in \Delta_\Theta$. Defining $r = -(1/\Theta_i) + (1/\Theta_j)$, it is easily seen that $(1/r)(-(1/\Theta_i)\, e^i + (1/\Theta_j)\, e^j) \in \Delta_\Theta$, for all $i \in L^-$ and $j \in L^+$. Hence, $q\bar{\zeta} \leqslant 0$ for all $q \in \Delta_\Theta$ implies, in particular, that $0 \geqslant -(\bar{\zeta}_i/\Theta_i) + (\bar{\zeta}_j/\Theta_j)$ for all $i \in L^-$ and $j \in L^+$. However, $\bar{\zeta} \in z^*(\bar{p})$ implies $(\bar{\zeta}_j/\Theta_j) \geqslant (\bar{\zeta}_i/\Theta_i)$ for $i \in L^-$ and $j \in L^+$. We conclude that $\bar{\zeta}_i/\Theta_i = \bar{\zeta}_j/\Theta_j$ for all $i \in L^-$ and $j \in L^+$. Hence, $\bar{\zeta} = k\Theta$ for some $k \in \mathbb{R}$. Because $\bar{\zeta} \in z^*(\bar{p})$, it holds that $\bar{\zeta}\Theta = 0$ and, hence, we conclude that $k = 0$ and, consequently, $\bar{\zeta} = 0$. □

Now, let $E \in \mathscr{F}$ be a generalized economy where the agents have smooth utility functions and let $\rho \in \mathbb{R}_+$ be such that the conditions in Theorem 4.1 are satisfied. Then it follows by weak monotonicity of the utility functions that $\mathrm{W}(E_\rho) \subseteq \mathrm{W}(E)$ and, hence, using Theorem 4.1 we see that the set of Walras allocations for E is non-empty.

Remark 4.2. The scope of Theorem 4.1 is not so large, in the sense that it will often be impossible to find a $\rho \in \mathbb{R}_+$ that satisfies the conditions of this theorem. The theorem merely provides sufficient conditions for existence, but these conditions are far from being necessary. However, at the moment we do not see how to improve upon the theorem. Trying to find a more general existence theorem is the subject of further research.

5. A characterization of the Walras utilities

In this section we provide a characterization of the vectors of utilities that correspond to strictly positive Walras equilibria, based on consistency and non-emptiness.

Since we only want to consider strictly positive Walras allocations and their utility vectors, we introduce the notation:

$$\mathrm{W}_{++}(E) = \mathrm{W}(E) \cap \mathbb{R}^{\ell |N|}_{++},$$

for all $E \in \mathscr{E}$. Furthermore, we no longer need condition (i) of smoothness, so we define $\mathscr{F}_{++}$ to be the family of all generalized economies in $\mathscr{E}$ where the agents have utility functions that satisfy condition (ii) of smoothness. Furthermore, let $\mathscr{G}_{++}$ be the family of generalized economies in $\mathscr{F}_{++}$ that allow for a strictly positive Walras allocation.

Let $E = \langle N; (w^i)_{i\in N}; (u^i)_{i\in N}; \Theta\rangle \in \mathscr{E}$. We define a function U_E: $A(E) \to \mathbb{R}^{N(E)}$ by $U_E((x^i)_{i\in N(E)}) = ((u^i(x^i))_{i\in N(E)})$ for every $x \in A(E)$. We denote

$$U(E) = \{U_E(x) \mid x \in A(E)\}.$$

A *u-solution* on $\mathscr{E}$ is a function ψ on $\mathscr{E}$ that assigns to each $E \in \mathscr{E}$ a subset $\psi(E)$ of $U(E)$. The *u-Walras correspondence* is defined by

$$u\mathrm{W}(E) = \{(u^i(x^i))_{i\in N(E)} \mid (x^i)_{i\in N(E)} \in \mathrm{W}_{++}(E)\},$$

for all $E \in \mathscr{E}$.

In the next lemma we show that we can enlighten utility vectors corresponding to strictly positive Walras allocations. Lemma 5.1 is a modified 'ancestors property' for the u-Walras correspondence (see Peleg et al. 1994).

Lemma 5.1. Let $E = \langle N; (w^i)_{i\in N}; (u^i)_{i\in N}; \Theta\rangle \in \mathscr{G}_{++}$, $x \in \mathrm{W}_{++}(E)$ and $p \in \Delta$ such that $x^i \in D^i(p)$ for each $i \in N$. Then there exists a generalized economy $\bar{E} = \langle N \cup \{0\}; (w^i)_{i\in N\cup\{0\}}; (u^i)_{i\in N\cup\{0\}}; \Theta\rangle \in \mathscr{G}_{++}$ with the property that $u\mathrm{W}(\bar{E}) = \{v\}$, where $v^0 = pw^0$ and $v^i = u^i(x^i)$ for all $i \in N$.

Proof. Pick an agent $0 \notin N$ and endow this agent with a utility function u^0 such that $u^0(y) = py$ for all $y \in \mathbb{R}^{\ell}$. Furthermore, the initial endowment of agent 0 is some $w^0 \in \mathbb{R}^{\ell}_{++}$. We enlarge the economy E by adding agent 0, so that we obtain the economy:

$$\bar{E} = \langle N \cup \{0\}; (w^i)_{i\in N\cup\{0\}}; (u^i)_{i\in N\cup\{0\}}; \Theta\rangle.$$

Note that $(y^i)_{i\in N\cup\{0\}} := (w^0, x^1, \ldots, x^n)$ is a Walras allocation for $\bar{E}$, because $y^i \in D^i(p)$ for each $i \in N \cup \{0\}$. This implies that $(pw^0, (u^i(x^i))_{i\in N}) \in u\mathrm{W}(\bar{E})$. To prove the other inclusion ($u\mathrm{W}(\bar{E}) \subseteq \{(pw^0, (u^i(x^i))_{i\in N})\}$), we first show that a price vector $p' \neq p$ does not support a strictly positive Walras allocation for economy $\bar{E}$. Notice that agent 0 has a linear utility function. This implies that for a price vector $p' \neq p$, an allocation $y \in D^0(p')$ has the property that $y_k = 0$ for at least one $k \in \{1, \ldots, l\}$. Hence, for each price vector $p' \neq p$, it holds that $D^0(p') \cap \mathbb{R}^{\ell}_{++} = \emptyset$. Now, the proof is concluded by noting that $U_{\bar{E}}(y) = U_{\bar{E}}(z)$ for all Walras allocations y and z for economy $\bar{E}$ that are supported by price vector p. □

In the proof of Lemma 5.1 we use a linear utility function. Clearly, this utility function does not satisfy condition (i) of smoothness. It is for this reason that we have to restrict to strictly positive Walras allocations.

In order to characterize the u-Walras correspondence, we need the following three axioms.

Axiom 1. A u-solution ψ on a family $\mathscr{E}_0 \subseteq \mathscr{E}$ satisfies *non-emptiness* (NEM) if $\psi(E) \neq \emptyset$ for every $E \in \mathscr{E}_0$.

Axiom 2. A u-solution ψ on a family $\mathscr{E}_0 \subseteq \mathscr{E}$ is *Pareto optimal* (PO) if for every $E \in \mathscr{E}_0$ and every $v \in \psi(E)$ all $x \in A(E)$ with $U_E(x) = v$ are efficient allocations. Note that this is equivalent to saying that for every $E \in \mathscr{E}_0$ and every $v \in \psi(E)$ it holds that there is no allocation $y \in A(E)$ such that $u^i(y) \geqslant v^i$ for all $i \in N(E)$ with strict inequality for at least one $i \in N(E)$.

Axiom 3. A u-solution ψ on a family $\mathscr{E}_0 \subseteq \mathscr{E}$ is *consistent* (CONS) if for all $E \in \mathscr{E}_0$ and all $v \in \psi(E)$ the following condition is satisfied. If $x \in A(E)$ is such that $U_E(x) = v$, then $E^{S,x} \in \mathscr{E}_0$ and $(v^i)_{i \in S} \in \psi(E^{S,x})$ for all $S \subseteq N(E)$, $S \neq \emptyset$.

Theorem 5.2. The u-Walras correspondence is the unique u-solution on $\mathscr{G}_{++}$ that satisfies NEM, PO, and CONS.

Proof. It is easily verified that the u-Walras correspondence satisfies NEM and PO on $\mathscr{G}_{++}$. Consistency of the u-Walras correspondence on $\mathscr{G}_{++}$ follows from the observation that if $x \in \mathrm{W}(E)$ and $y \in A(E)$ such that $U_E(y) = U_E(x)$, then $y \in \mathrm{W}(E)$ (this is a well-known property of the Walras allocations; see, for example, the non-discrimination axiom in Nagahisa, 1991).

Now, let ψ be a u-solution on $\mathscr{G}_{++}$ that satisfies the three axioms and let $E \in \mathscr{G}_{++}$. If $|N(E)| = 1$, then $|A(E)| = |U(E)| = 1$ and, by NEM, $u\mathrm{W}(E) = \psi(E) = U(E)$. If $|N(E)| \geqslant 2$, let $v \in \psi(E)$ and $x \in A(E)$ such that $U_E(x) = v$. Then, by PO of ψ, x is an efficient allocation. Also, $x^i \in A(E^{\{i\},x})$ and, by CONS of ψ, $U_{E^{\{i\},x}}(x^i) = v^i \in \psi(E^{\{i\},x})$ for all $i \in N(E)$. Because $E^{\{i\},x}$ is a one-agent economy, this implies that $v^i \in u\mathrm{W}(E^{\{i\},x})$ and, hence, $x^i \in \mathrm{W}_{++}(E^{\{i\},x})$ for all $i \in N(E)$. Now, Lemma 3.4 implies that $x \in \mathrm{W}_{++}(E)$. Hence, $v = U_E(x)$ is in $u\mathrm{W}(E)$. Thus, $\psi(E) \subseteq u\mathrm{W}(E)$ for all $E \in \mathscr{G}_{++}$.

We shall use Lemma 5.1 to prove that $u\mathrm{W}(E) \subseteq \psi(E)$ for all $E \in \mathscr{G}_{++}$. Let $E \in \mathscr{G}_{++}$ and let $v \in u\mathrm{W}(E)$. Choose $x \in \mathrm{W}_{++}(E)$ such that $v = U_E(x)$. Furthermore, let $p \in \Delta$ be a price vector such that $x^i \in D^i(p)$ for each $i \in N(E)$. Consider the economy $\bar{E}$ that we defined in Lemma 5.1. For this economy, we have $u\mathrm{W}(\bar{E}) = \{\bar{v}\}$, where $\bar{v}^0 = pw^0$ and $\bar{v}^i = u^i(x^i)$ for all $i \in N(E)$. Since $\psi(\bar{E}) \subseteq u\mathrm{W}(\bar{E})$, it follows by NEM of ψ that $\psi(\bar{E}) = \{\bar{v}\}$. Hence, by CONS of ψ, we know that v is an element of ψ of the reduced economy of $\bar{E}$ with respect to $N(E)$ and $(w^0,(x^i)_{i \in N(E)})$. This reduced economy is the economy E. Hence, we conclude that $v \in \psi(E)$. □

6. Economies with free disposal of commodities

If we assume that commodities are freely disposable, then we have to consider for a generalized economy $E = \langle N; (w^i)_{i \in N}; (u^i)_{i \in N}; \Theta \rangle$ the set of allocations:

$$A_f(E) = \left\{ (x^i)_{i \in N} \mid x^i \in \mathbb{R}^l_+ \text{ for all } i \in N, \sum_{i \in N} x^i \leqslant \sum_{i \in N} w^i + \Theta \right\},$$

and, correspondingly, a *solution* on a family $\mathscr{E}_0 \subseteq \mathscr{E}$ is a function ϕ that assigns to each economy $E \in \mathscr{E}_0$, a subset $\phi(E)$ of $A_f(E)$.

Pareto-optimal allocations and Walras allocations are defined with respect to the set of allocations $A_f(E)$ and, correspondingly, we denote these solutions by $PO_f(E)$ and $W_f(E)$, respectively. The definitions of the Pareto-optimal solution and the Walras correspondence are straightforward generalizations of the definitions we provided in Section 2 and therefore we will not repeat them.

A striking difference between the approach without free disposal and the approach with free disposal is that Walras allocations need no longer be Pareto optimal if there is free disposal. This is shown in the following example.

Example 6.1. Let E be a two-agent economy with two goods. The initial endowment of agent 1 is $w^1 = (1, 1)$, i.e. agent 1 is endowed with 1 unit of the first good and one unit of the second good, and the initial endowment of agent 2 is $w^2 = (2, 1)$. The utility functions of agents 1 and 2 are given by $u^1(x_1, x_2) = 2x_1 + x_2$ and $u^2(x_1, x_2) = x_1 + 2x_2$, respectively. The net trade vector for the economy is $\Theta = (-1, 2)$. A Walras allocation of this economy is given by $\hat{x}^1 = (2, 0)$ and $\hat{x}^2 = (0, 3)$. The corresponding price vector is $p = (\frac{1}{2}, \frac{1}{2})$. The allocation $y \in A(E)$ defined by $y^1 = (2, 0)$ and $y^2 = (0, 4)$ satisfies $u^1(y^1) = u^1(\hat{x}^1)$ and $u^2(y^2) > u^2(\hat{x}^2)$. Hence, the Walras allocation $\hat{x}$ is not efficient.

Clearly, the non-efficiency of the Walras allocation in Example 6.1 is caused by the fact that, in this allocation, supply is not equal to demand (it is in the set $A_f(E) \setminus A(E)$), because it follows from our previous results that a Walras allocation in which supply equals demand is efficient. It would be interesting to find out for which economies there exists a Walras allocation in which supply equals demand. A partial answer to this question is given in the next proposition.

Proposition 6.2. Let $E \in \mathscr{F}$ be a generalized economy, $x \in A_f(E)$ a Walras allocation, and let $p \in \Delta$ be a price vector such that $x^i \in D^i(p)$ for each $i \in N$. Then the following two assertions hold:

(i) $p\Theta \geqslant 0$.

(ii) *If $p\Theta = 0$, then x is Pareto optimal in E and, moreover, E has a Walras allocation in which supply equals demand.*

Proof. Let $\varepsilon = \sum_{i \in N} w^i + \Theta - \sum_{i \in N} x^i$. Then $\varepsilon \in \mathbb{R}^l_+$ and, consequently, $p\varepsilon \geqslant 0$. Further, by local non-satiation $pw^i = px^i$ for all $i \in N$, and, consequently, $p\varepsilon = p\Theta$. This proves part (i). To prove part (ii), suppose $p\Theta = 0$ and let $j \in N$. Then $x^j, x^j + \varepsilon \in \mathbb{R}^l_{++}$ and, because of weak monotonicity, $u^j(x^j + \varepsilon) \geqslant u^j(x^j)$. However, because $p(x^j + \varepsilon) = px^j$, it holds that $u^j(x^j) \geqslant u^j(x^j + \varepsilon)$. Thus, $u^j(x^j + \varepsilon) = u^j(x^j)$ and, hence, $x^j + \varepsilon \in D^j(p)$. So, we found a Walras allocation, $x_\varepsilon = (x^j + \varepsilon, (x^i)_{i \in N \setminus \{j\}})$, in which supply equals demand. Now, Pareto optimality of x follows by noting that $u^i(x^i_\varepsilon) = u^i(x^i)$ for each $i \in N$ and that x_ε is Pareto optimal. □

Note that for an ordinary economy ($\Theta = 0$) and all its reduced economies with respect to some Walras allocation the condition in Proposition 6.2 (ii) is satisfied and, hence, these economies have a Walras allocation in which supply equals demand. Of course, we do not need Proposition 6.2 to see this, because it follows using consistency of the Walras correspondence.

Since Walras allocations need not be Pareto optimal, we cannot use the axiom PO(2) in order to characterize the Walras allocations. We have to adapt the axiom PO(2) and we do it in the following way.

Let $E = \langle N; (w^i)_{i \in N}; (u^i)_{i \in N}; \Theta \rangle$ be a generalized economy and let $x \in A_f(E)$. Then x is *self-efficient* if x is efficient in the economy $\langle N; (x^i)_{i \in N}; (u^i)_{i \in N}; 0 \rangle$. Hence, an allocation x is self-efficient if the total amount of commodities that is allocated to the agents ($\Sigma_{i \in N} x^i$) cannot be re-allocated in such a way that all agents are at least as well off and one agent is strictly better off. Clearly, Walras allocations are self-efficient.

Let $\mathscr{E}_0$ be a family of generalized economies. A solution ϕ on $\mathscr{E}_0$ satisfies *self-efficiency for two-agent generalized economies* (SE(2)) if for every two-agent generalized economy $E \in \mathscr{E}_0$ all $x \in \phi(E)$ are self-efficient.

Now we modify the definition of converse consistency to fit the situation in which commodities are freely disposable. Let $\mathscr{E}_0$ be a family of generalized economies. A solution ϕ on $\mathscr{E}_0$ satisfies *converse consistency* (COCONS) if for every $E \in \mathscr{E}_0$ with at least two agents ($|N(E)| \geq 2$) and for every $x \in A_f(E)$ that is self-efficient the following condition is satisfied. If for every $S \subseteq N(E)$, $S \notin \{\emptyset, N(E)\}$, it holds that $E^{S,x} \in \mathscr{E}_0$ and $x^S \in \phi(E^{S,x})$, then $x \in \phi(E)$.

The following result is easily obtained by making slight adaptations in the proof of Theorem 3.6.

Theorem 6.3. The Walras correspondence is the unique solution on $\mathscr{F}$ that satisfies NMEX(1), SE(2), CONS and COCONS.

7. Future research

There are several questions in relation to this paper that are open for future research. We have mentioned already that our axiomatization of the Walras correspondence depends on smoothness of the utility functions. Characterizing the Walras correspondence for economies with more general utility functions remains an open problem that might require a different approach. There is also much material for future research in the area of existence. The existence theorem presented in Section 4 is not very sharp and it will be interesting to try and find an existence result that is more widely applicable.

Other ideas for future research are in the area of extending the type of economies taken into consideration. For example, instead of dealing with pure exchange economies only, one could allow for production to take place. A study of (private goods) production economies from the viewpoint of consistency might provide new insights. Consistency in the context of public good economies (with or without public goods production) is studied by van den Nouweland et al. (1995).

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